Paraglider: Mathematical model and control

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Paraglider: Mathematical Model and Control

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The design of remotely controlled and unmanned aerial vehicles is an important direction in modern aircraft development [1]. A promising aircraft of this type is a paraglider.

In this paper, we suggest a mathematical model of paraglider's planar longitudinal motion aimed at the study of paraglider dynamics and the synthesis of its automatic control. The mechanical model of the vehicle represents a single rigid body with three degrees of freedom that consists of a wing (sail) and the gondola. The sail and the gondola are connected by cords, which are modeled by absolutely rigid rods. A driver (propeller) producing a thrust is mounted on the gondola. We construct a control law for the thrust that stabilizes the flight of the vehicle at a preset altitude.

1. MATHEMATICAL MODEL OF PARAGLIDER'S LONGITUDINAL MOTION

A schematic view of the paraglider is presented in Fig. 1. The wing (sail) is depicted as a straight-line segment centered at the point A. Assume that the lift $P = C_y\frac{1}{2}\rho V_A^2 S$ and the frontal drag $Q = C_x\frac{1}{2}\rho V_A^2 S$ are both applied at A. Here, C_y and C_x are the lift and front drag coefficients [2, 3], ρ is the air density, V_A is the velocity of A relative to the incoming flow, and S is the surface area of the sail. Of course, the application point of the forces P and Q varies with the velocity and orientation of the sail. However, we ignore this circumstance, because the torque of these forces relative to the paraglider's center of mass C depends "weakly" on the location of their application point if their arm (the distance between the paraglider's center of mass and the sail) is large as compared to the size of the sail.

Denote by G the center of mass of the gondola and by $GA = L$ the distance between G and A. Let m_1 be the mass of the gondola, m_2 be the mass of the sail, and $m_1 + m_2 = M$. Then $GC = l_1 = \frac{m_2 L}{M}$ and $CA = L - l_1 = l_2$. If $m_2 \ll m_1$, then $l_1 \ll l_2$; i.e., the center of mass of the entire vehicle is close to the gondola. Let J denote the moment of inertia of the vehicle relative to C and g be the acceleration of gravity. If the lengths of the cords are different, then the angle between the sail and the line AG is not right. The angle between the perpendicular to the sail and AG is denoted by σ (see Fig. 1). The lift of the sail can be controlled by turning it in the longitudinal plane through a constant angle σ.

We introduce a coordinate system XOY fixed with respect to the ground surface with OX and OY being its horizontal and vertical axes, respectively. Let x and y be the coordinates of the paraglider's center of mass C, V be the velocity of the point C, and θ is the angle between the velocity $\mathbf{V}$ of C and the positive OX axis. The pitch angle between the vertical axis OY and the line GA is measured counterclockwise and is denoted by ϑ (see Fig. 1); $\dot{\vartheta} = \omega$.

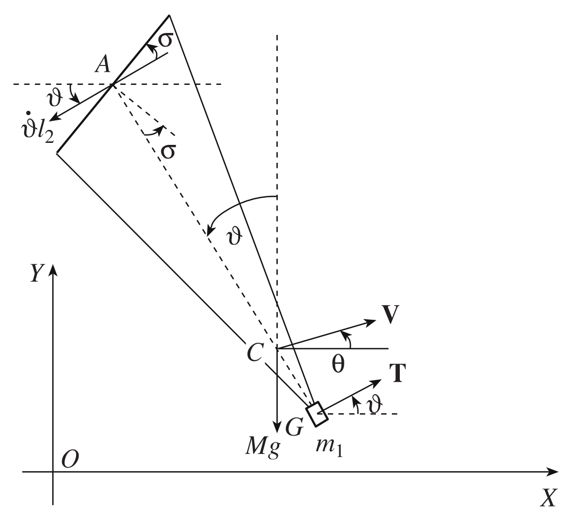

Fig. 1. Schematic view of the paraglider.

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The velocities $\mathbf{V_A}$ and $\mathbf{V_G}$ of A and G, respectively, are the sum of the velocity of C and the velocities of these points relative to C. They are calculated by the formulas

$$V_A = \sqrt{V^2 + \omega^2 l_2^2 - 2V\omega l_2 \cos(\vartheta - \theta)},$$

$$V_G = \sqrt{V^2 + \omega^2 l_1^2 + 2V\omega l_1 \cos(\vartheta - \theta)}.$$

The angle β_A between $\mathbf{V}$ and $\mathbf{V_A}$ is determined using the law of sines: $\sin\beta_A = \dfrac{\omega l_2 \sin(\vartheta - \theta)}{V_A}$. The angle β_G between the velocities $\boldsymbol{\omega}\mathbf{l_1}$ and $\mathbf{V_G}$ is given by the formula $\sin\beta_G = \dfrac{V\sin(\vartheta - \theta)}{V_G}$.

In the linear approximation with respect to the angle of attack α, the coefficient C_y can be represented as $C_y = C_y^\alpha \alpha$, where C_y^α (like C_x) is a constant. To take into account the nonlinear dependence of the lift and frontal drag coefficients on the angle of attack, we have to know the polar relating C_y to C_x for various angles of attack [2, 3].

The equations of motion of C in projections onto the tangent to its trajectory and the normal can be written as

$$\begin{aligned} M\dot{V} &= -Mg\sin\theta + T\cos(\vartheta - \theta) \\ &+ C_y^\alpha(\vartheta - \theta + \beta_A + \sigma)\frac{1}{2}\rho V_A^2 S \sin\beta_A \\ &- C_x \frac{1}{2}\rho V_A^2 S\cos\beta_A - C_{xG}\frac{1}{2}\rho V_G^2 S\cos(\vartheta - \theta - \beta_G) \\ &+ R_y \sin\theta - R_x\cos\theta, \end{aligned} \quad (1)$$

$$\begin{aligned} MV\dot{\theta} &= -Mg\cos\theta + T\sin(\vartheta - \theta) \\ &+ C_y^\alpha(\vartheta - \theta + \beta_A + \sigma)\frac{1}{2}\rho V_A^2 S \cos\beta_A \\ &+ C_x \frac{1}{2}\rho V_A^2 S\sin\beta_A - C_{xG}\frac{1}{2}\rho V_G^2 S\sin(\vartheta - \theta - \beta_G) \\ &+ R_y \cos\theta + R_x\sin\theta, \end{aligned} \quad (2)$$

respectively. Here, the first terms on the right-hand sides of (1) and (2) describe the projections of the vehicle's gravity force onto the corresponding axes; the second terms describe the projections of the thrust T (assuming that it is applied at G and is perpendicular to the line AG); the third terms, the projections of the lift ($\vartheta - \theta + \beta_A + \sigma$ is the angle of attack α); the fourth and fifth terms, the projections of the frontal drag applied to the sail and the gondola, respectively (C_{xG} = const is the frontal drag coefficient for the gondola); R_y is the vertical component of the support reaction, which is directed upward (positive) when the gondola wheels are rolled on the ground at the take-off run of the paraglider; and R_x is the horizontal drag of the gondola rolling on the ground. After the paraglider takes off the ground, $R_x = R_y = 0$.

The equations of motion of the vehicle about its center of mass are

$$\begin{aligned} J\ddot{\vartheta} &= -C_y^\alpha \frac{1}{2}\rho V_A^2 S(\vartheta - \theta + \beta_A + \sigma) l_2 \sin(\vartheta - \theta + \beta_A) \\ &+ C_x\frac{1}{2}\rho V_A^2 S l_2\cos(\vartheta - \theta + \beta_A) \\ &- C_{xG}\frac{1}{2}\rho V_G^2 S l_1 \cos\beta_G + T l_1 + (R_y\sin\vartheta - R_x\cos\vartheta) l_1. \end{aligned} \quad (3)$$

The motion of the paraglider's center of mass is governed by the obvious kinematic relations

$$\dot{x} = V\cos\theta, \quad \dot{y} = V\sin\theta.$$

When accelerating during the take-off run, the paraglider moves on the ground. In the course of flight, it flies above the ground. Therefore, the vertical coordinate h of G, which is equal to $y - l_1\cos\vartheta$, is subject to the unilateral constraint $h \geq 0$. For ground motion, $h \equiv 0$. Differentiating this identity twice and substituting into it the expressions for $\dot{V}$, $\dot{\theta}$, and $\ddot{\vartheta}$ from Eqs. (1)–(3), we can find R_y (if R_x is given). If the support reaction is positive, then substituting it into (1)–(3) yields the equations of motion for the gondola rolling on the ground. When the reaction reverses its sign, the vehicle takes off the ground.

2. STEADY-STATE FLIGHT REGIMES

If T = const, then, using dynamic equations (1)–(3), we can find the steady-state flight regime under which $V \equiv$ const, $\vartheta \equiv$ const, and $\theta \equiv$ const. In this regime, the paraglider moves uniformly and progressively along a straight line making an angle θ with the OX axis. Substituting $\dot{V} = 0$ and $\omega = \dot{\theta} = 0$ into differential equations (1)–(3) yields three algebraic equations relating four unknowns: T, ϑ, θ, and V. These nonlinear equations define a one-parameter family of steady-state regimes, which can be constructed numerically. It is convenient to use θ as a parameter in the numerical study. Then each given (reasonable) value of θ is associated with some values of ϑ, V, and T. These steady-state values include $\theta = 0$ and the corresponding values of ϑ, V, and T, with the last value denoted by T_*. In other words, the steady-state regimes include a horizontal flight at $T = T_*$ = const. For thrust values other than T_*, the paraglider in a steady-state regime follows an inclined trajectory. Therefore, the velocity of a horizontal flight cannot be changed by varying the thrust. Setting up variational equations for steady-state regimes, we can analyze their stability. Steady-state regimes with highly inclined trajectories are unstable.

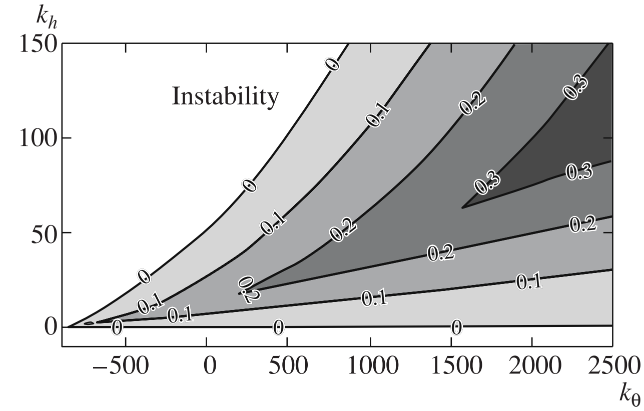

Fig. 2. Domains of asymptotic stability and stability with a margin of 0.1, 0.2, or 0.3 in the plane of k_h and k_θ.

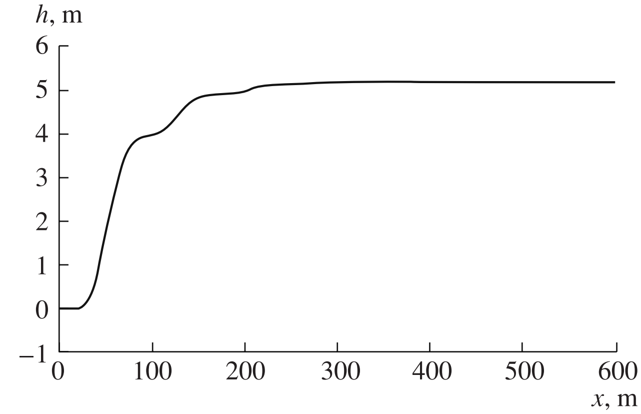

Fig. 3. Paraglider take-off and flight at the constant altitude $h = h(x)$.

3. STABILIZATION OF THE FLIGHT ALTITUDE

At small flight altitudes, the dependence of ρ on altitude can be neglected. Then the motion of the paraglider is independent of its altitude above the ground. In other words, the coordinate y is a cyclic variable. Therefore, the horizontal uncontrolled motion of the vehicle (at $T = T_* =$ const) is indifferent to y and, hence, is not asymptotically stable with respect to the flight altitude. A flight at the desired altitude can be stabilized by controlling the thrust. A stabilizing control is constructed in the form of feedback with respect to the deviation of the gondola's flight altitude from the desired (preset) value and with respect to θ:

$$T = T_s - k_h(h - h_d) - k_\theta \theta. \quad (4)$$

Here, T_s = const is a given thrust equal or close to T_*, h_d is the desired flight altitude of G above the ground, and k_h and k_θ are constant feedback factors.

Under control (4), system (1)–(3) exhibits a steady-state regime of motion: $V \equiv$ const, $\theta \equiv 0$, $\vartheta \equiv$ const, $h \equiv$ const, and $T \equiv T_*$. The steady-state flight altitude h is determined by the equality

$$h = h_d + \frac{T_s - T_*}{k_h}. \quad (5)$$

It follows that the error $\Delta h = |h - h_d|$ in the preset altitude h_d is smaller when $T_s - T_*$ is closer to zero and (or) when the feedback factor k_h is higher. However, if k_h is too high, the steady-state regime may become unstable [4].

The admissible thrust values are bounded above by a certain value T_m, and the thrust cannot be negative. Therefore, instead of (4), we consider the feedback

$$T = \begin{cases} T_m \ \text{if} \ T_s - k_h(h - h_d) - k_\theta\theta \ge T_m \\ T_s - k_h(h - h_d) - k_\theta\theta \\ \text{if} \ 0 \le T_s - k_h(h - h_d) - k_\theta\theta \le T_m \\ 0 \ \text{if} \ T_s - k_h(h - h_d) - k_\theta\theta \le 0. \end{cases} \quad (6)$$

4. NUMERICAL STUDY

The numerical study was performed for some hypothetical values of the paraglider parameters.

Figure 2 shows the domain of asymptotic stability constructed in the plane of k_h and k_θ. Inspection of the figure reveals that k_h is bounded above for each value of k_θ. This figure also displays the boundaries of the stability domains at a given stability margin. These are the domains in which $\mathrm{Re}\lambda_i \le 0.1, 0.2, 0.3$ ($i = 1, 2, 3, 4, 5$) or, in other words, in which the eigenvalue nearest to the imaginary axis lies to the left of it at a distance of 0.1, 0.2, or 0.3.

The code for solving system (1)–(3), (6) was written in Matlab. It includes the possibility of flight animation and provides the opportunity to analyze various regimes of paraglider motion. Figure 3 shows the trajectory of a paraglider taking off and flying at the constant altitude $h = 5.2$ m. On the acceleration segment of about 20 m, the gondola moves on the ground. The height $h_d = 5$ m is specified in (6). The static error $\Delta h = 0.2$ m can be reduced by increasing k_h. However, as this coefficient grows, the transitional process may become oscillatory. The oscillations can be suppressed by increasing k_θ.

Note that, when the initial velocity $V(0)$ is low, the angle of attack α ($\alpha = \vartheta - \theta + \beta_A + \sigma$) at the beginning of the motion takes "large" values for which the linearization of C_y with respect to $C_y = C_y^\alpha \alpha$ is incorrect. The mathematical model designed above is correct for sufficiently high initial velocities of the vehicle.

The flight of the paraglider at a preset altitude can be stabilized by applying a feedback control with respect to the altitude and its derivative:

$$T = \begin{cases} T_m \text{ if } T_s - k_h(h-h_d) - k_{\dot{h}}\dot{h} \geq T_m \\ T_s - k_h(h-h_d) - k_{\dot{h}}\dot{h} \\ \text{if } 0 \leq T_s - k_h(h-h_d) - k_{\dot{h}}\dot{h} \leq T_m \\ 0 \text{ if } T_s - k_h(h-h_d) - k_{\dot{h}}\dot{h} \leq 0. \end{cases}$$

Note that the paraglider's take-off and landing trajectories can be controlled by specifying h_d as a function of time or distance.

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